Metric Spaces

Definition of Metric Spaces

Definition: Metric Spaces

Suppose X is a set and d is a real function defined on the Cartesian product $X \times X$

Then, d is called a metric on X if and only if for each $a, b \in X$ the function

$$d : X \times X \to [0, \infty)$$

has the following properties $\forall\, a, b, c \in X$

(M1) Positive Property: $d(a,b) \geq 0$

(M2) $d(a,b) = 0 \iff a = b$

(M3) Symmetric property: $d(a,b) = d(b,a)$

(M4) Triangle Inequality $d(a,b) \leq d(a,c) + d(c,b)$

The pair of objects is called a metric space

Rearrangement of Triangle Inequality

Theorem Rearrangement of Triangle inequality

Suppose (X,d) is a metric space. Then $\forall\, a, b, c \in X$,

$$|d(a,b) - d(b,c)| \leq d(a,c)$$

Proof: By triangle inequality (M4)

$$d(a,b) \leq d(a,c) + d(c,b) \qquad (*1)$$

Similarly applying triangle inequality for $d(b,c)$

$$d(b,c) = d(a,c) + d(a,b)$$

Rearranging $(*1)$ and $(*2)$ gives

$$d(a,b) - d(b,c) \leq d(a,c) \quad \text{and} \quad d(b,c) - d(a,c) \leq d(a,c)$$

$$\Rightarrow \quad |d(a,b) - d(b,c)| \leq d(a,c)$$

by definition of absolute value

$\blacksquare$

Useful tip

For property M2, we can split the biconditional into 2 if statements

$$\boxed{\begin{aligned} a=b &\Rightarrow d(a,b)=0 \\ d(a,b)=0 &\Rightarrow a=b \end{aligned}}$$

and we can prove individually or take contrapositives

Examples of Metric Spaces

Standard Metric on $\mathbb{R}$

Theorem Standard metric on $\mathbb{R}$

$(\mathbb{R}, d)$ is a metric space where d is the function defined by

$$d: \mathbb{R}\times\mathbb{R} \to [0,\infty)$$

$$d(x,y) = |x-y| \quad \forall x,y \in \mathbb{R}$$

Proof: To prove d is a metric, we need to verify axioms M1-M4

(M1): $d(x,y) \geq 0$ by definition of absolute value

(M2):
$$\begin{aligned} d(x,y)=0 &\iff |x-y|=0 \\ &\iff \sqrt{(x-y)^2}=0 \\ &\iff (x-y)^2=0 \\ &\iff x=y \end{aligned}$$

(M3) $d(x,y) = |x-y| = |y-x| = d(y,x)$

(M4) $\forall x,y,z \in \mathbb{R}$, $d(x,z) = |x-z| = |x-y+y-z|$ (add 0 trick)

$$\begin{aligned} &\leq |x-y| + |y-z| \\ &= d(x,y) + d(y,z) \end{aligned}$$

$$\Rightarrow d(x,z) = d(x,y) + d(y,z)$$

$\blacksquare$

Generalised metric on $\mathbb{R}^N$

Theorem Generalised metric is a metric

Let $X=\mathbb{R}^N$ and $d_p:\mathbb{R}^N \times \mathbb{R}^N \to [0,\infty)$ defined by

$$d_p(x,y) = \left(\sum_{i=1}^{n} |x_i - y_i|^p\right)^{1/p} \quad \text{for } p\in\mathbb{N} \quad \forall x, y \in \mathbb{N}$$

Then $(\mathbb{R}, d_p)$ is a metric space

Proof: Verifying M4 Triangle Inequality

$$d_p(x,z) = \left(\sum_{i=1}^{N} |x_i - z_i|^p\right)^{1/p}$$

$$d_p(y,z) = \left(\sum_{i=1}^{N} |y_i - z_i|^p\right)^{1/p}$$

Define $a_i = x_i - z_i$, $b_i = y_i - z_i \implies a_i + b_i = x_i - y_i$.

Therefore

$$d_p(x,y) = \left(\sum_{i=1}^{N} |x_i - y_i|^p\right)^{1/p} = \left(\sum_{i=1}^{N} |a_i + b_i|^p\right)^{1/p}$$

$$d_p(y,z) = \left(\sum_{i=1}^{N} |b_i|^p\right)^{1/p} \qquad d_p(x,z) = \left(\sum_{i=1}^{N} |a_i|^p\right)^{1/p}$$

and to satisfy triangle inequality, we need

$$d_p(x,y) \leq d_p(x,z) + d_p(y,z)$$

$\implies$

$$\left(\sum |a_i + b_i|^p\right)^{1/p} \leq \left(\sum |a_i|^p\right)^{1/p} + \left(\sum |b_i|^p\right)^{1/p}$$

Which is just the Minkowski inequality

$\blacksquare$

Max metric on $\mathbb{R}^N$

Theorem

$$X = \mathbb{R}^N = \{\underline{x} = (x_1, x_2, \dots, x_n) : x_i \in \mathbb{R},\ 1 \le i \le n\}$$

$$d_\infty(\underline{x}, \underline{y}) = \max\{|x_i - y_i| : 1 \le i \le N\}$$

Then $(\mathbb{R}^N, d_\infty)$ is a metric space

Proof: Just checking (M4), triangle inequality,

$$|x_i - z_i| \le |x_i - y_i| + |y_i - z_i| \qquad \text{(0 trick)}$$

$$\le \max_{1 \le i \le n} |x_i - y_i| + \max_{1 \le i \le n} |y_i - z_i| \qquad \forall i \in \{1, .., N\}$$

$$\Rightarrow \max_{1 \le i \le n} |x_i - z_i| \le \max_{1 \le i \le n} |x_i - y_i| + \max_{1 \le i \le n} |y_i - z_i| \qquad \text{(since above true } \forall i\text{)}$$

$$\Rightarrow d_\infty(x, z) \le d_\infty(x, y) + d_\infty(y, z)$$

■

Discrete metric space

Theorem

(X, d_0) is a metric space where

$$d_0(x, y) = \begin{cases} 1 & x \ne y \\ 0 & x = y \end{cases}$$

Proof: Showing triangle inequality (M4)

Take any $x, y, z \in X$.

if $x = y$, then $d(x, y) = 0 \le \underbrace{d(x, z) + d(z, y)}_{0,\,1 \text{ or } 2}$

if $x \ne y$, then $z \ne x$ or $z \ne y$ (otherwise $z = x$ and $z = y \Rightarrow x = y$)

$$\underbrace{d(x, z) + d(y, z)}_{1 \text{ or } 2} \ge 1$$

$$\Rightarrow d(x, y) = 1 \le d(x, z) + d(y, z)$$

■

Canonical metrics on $\mathbb{R}^N$

Definition Canonical metrics on $\mathbb{R}^N$

Consider $X = \mathbb{R}^N$ and $d_1, d_2, d_\infty : \mathbb{R}^N \times \mathbb{R}^N \to [0,\infty)$. Then $\forall \underline{x}, \underline{y} \in \mathbb{R}^N$

$$d_1(\underline{x},\underline{y}) = \sum_{i=1}^{N} |x_i - y_i|$$

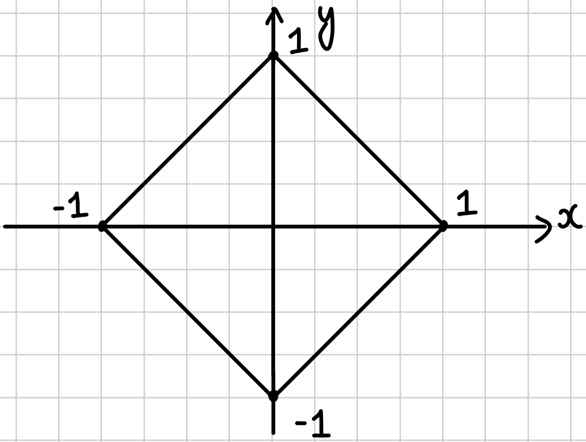
$$d_2(\underline{x},\underline{y}) = \left(\sum_{i=1}^{N} |x_i - y_i|^2 \right)^{1/2}$$

$$d_\infty(\underline{x},\underline{y}) = \max\{ |\underline{x}_i - \underline{y}_i| : 1 \le i \le N \}$$

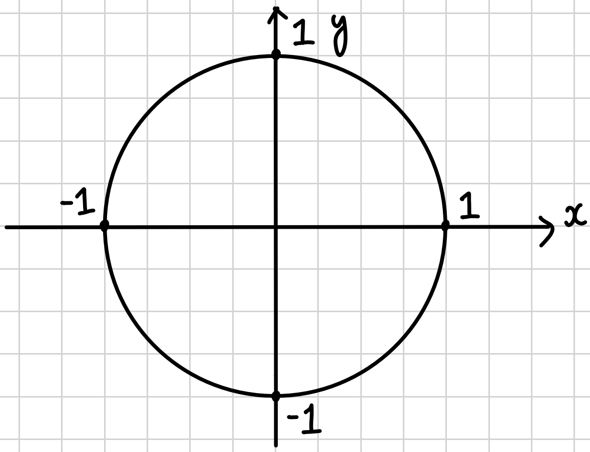
Unit circles in $\mathbb{R}^N$

Work in $\mathbb{R}^2$ and draw graphs

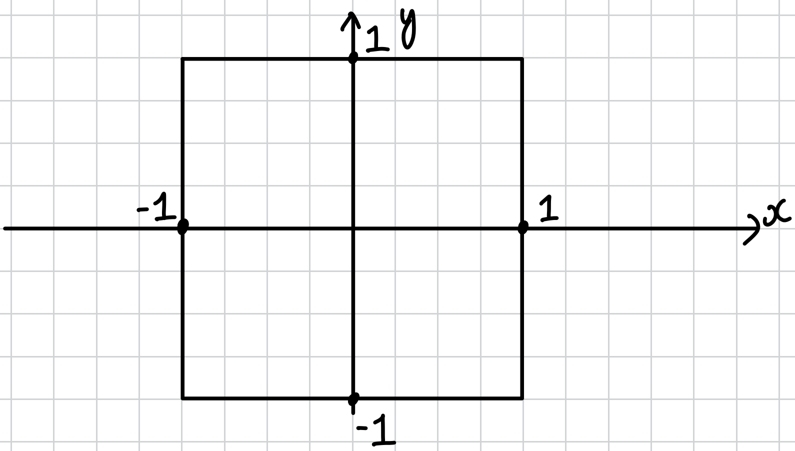
1) $A = \{(x,y) \in \mathbb{R}^2 : d_1(x,y) = 1\}$



2) $B = \{(x,y) \in \mathbb{R}^2 \mid d_2(x,y) = 1\}$



3) $C = \{(x,y) \in \mathbb{R}^2 \mid d_\infty(x,y) = 1\}$



Review of Real Analysis

Boundedness of sets

Definition: Bounded above and Upperbound

A subset S of an ordered field $\mathbb{K}$ is said to be bounded above if $\exists$ a $b \in \mathbb{K}$ such that

$$x \leq b \qquad \forall x \in S$$

Such a constant b is called the upperbound

Definition Bounded below and Lower bound

A subset S of an ordered field $\mathbb{K}$ is said to be bounded below if $\exists$ a $a \in \mathbb{K}$ such that

$$x \geq a \qquad \forall x \in S$$

Such a constant a is called the lowerbound

Definition Bounded

A subset S of an ordered field $\mathbb{K}$ is said to be bounded if it is both bounded above AND below

$\exists$ $a, b \in \mathbb{K}$ such that

$$a \leq x \leq b \quad \forall x \in S$$

Examples of Boundedness

1) Consider the following set:

$$S_1 = \{x \in \mathbb{K} : a < x < b\} = (a, b)$$

- Here S_1 is bounded below with a lower bound of a (or anything less than a)
- Here S_1 is bounded above with an upper bound of b (or anything bigger than b)

2) Consider the set

$$S_2 = \{x \in \mathbb{K} : a < x\} = (a, \infty)$$

- Here S_2 is bounded below with a lower bound of a (or anything smaller than a)
- Here S_2 is not bounded above

3) Consider another example:

$$S_3 = \{x \in \mathbb{K} : x < b\} = (-\infty, b)$$

- Here S_3 is bounded above with an upper bound of b (or anything bigger than b)
- Here S_3 is not bounded below

4) $\mathbb{K}$ itself is neither bounded above or below

Maximum and Minimum Element

Definition Maximum/Minimal Element

Suppose $S \subseteq \mathbb{K}$

We say that S has a maximal/maximum element say

$$\max(S)$$

if it contains an element larger than all other elements i.e. it is an upperbound of S AND an element of S

$$\max(S) \in S \quad \text{and} \quad x \leq \max(S) \qquad \forall s \in S$$

Definition Minimum/Minimal Element

Suppose $S \subseteq \mathbb{K}$

We say that S has a minimal/minimum element say

$$\min(S)$$

if it contains an element smaller than all other elements i.e. it is a lowerbound of S AND an element of S

$$\min(S) \in S \quad \text{and} \quad x \geq \min(S) \qquad \forall s \in S$$

Note Unlike upper and lower bounds, maximum and minimum elements ARE UNIQUE if they exist at all.

To see this, suppose b_1 and b_2 are maximal elements of S. Then they are both elements of S and both upperbounds of S. This means that

$$\underbrace{b_1}_{\text{element of } S} \leq \underbrace{b_2}_{\text{upperbound for } S}$$

$$\underbrace{b_2}_{\text{element of } S} \leq \underbrace{b_1}_{\text{upperbound for } S}$$

and therefore we can say $\boxed{b_1 = b_2}$

(Similar argument for minimal element)

Axiom of Completeness

The axiom of completeness is an important property of the real numbers that encapsulates the idea of NO GAPS on the real line

Axiom Axiom of Completeness

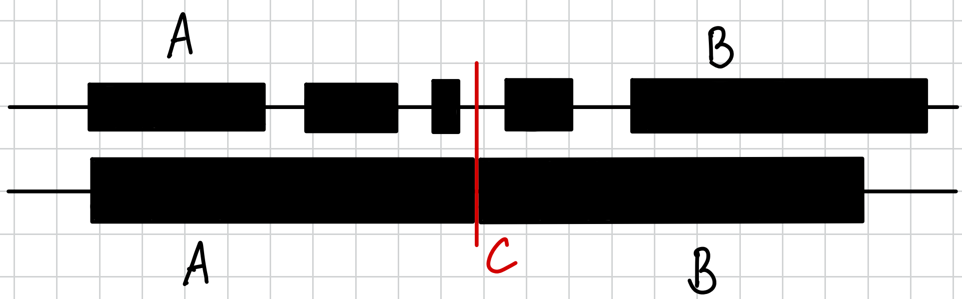
Suppose A and B are non-empty subsets of $\mathbb{R}$

$$A \neq \emptyset, \quad B \neq \emptyset, \quad A, B \subseteq \mathbb{R}$$

with the property that $\forall a, b,\ a \in A$ and $b \in B \implies a \leq b$

Then $\exists c \in \mathbb{R}$ such that $\forall a \in A,\ b \in B$

$$a \leq c \leq b$$



In terms of bounds,

- Every element of A is a lowerbound for B
- Every element of B is a upperbound for A
- This property of $\mathbb{R}$ is called completeness

Infimum and Supremum

We can use axiom of completeness to define infifum and supremum

Theorem: Infifum and Supremum

Suppose S is a non-empty subsets of $\mathbb{R}$, $S \subseteq \mathbb{R}$, $S \neq \emptyset$

1) Supremum: If S is bounded above then there is a minimal/least upperbound called supremum

2) Infifum: If S is bounded below then there is a maximal/greatest lowerbound called infifum

So

$\boxed{\text{bounded above} \implies \sup(S) \text{ exists}}$ $\boxed{\text{bounded below} \implies \inf(S) \text{ exists}}$

Proof:

1) Suppose S is bounded above

Let B be the set of all upperbounds and since S is bounded above, $B \neq \emptyset$

$$x \in S \text{ and } b \in B \implies x \leq b$$

(by definition of upper bound)

By the axiom of completeness,

$\exists c \in \mathbb{R}$ such that $\forall x \in S$ and $b \in B$, $x \leq c \leq b$

$\implies x \leq c$ AND $c \leq b$
(1) (2)

We can draw the following conclusions

(1): $x \leq c \quad \forall x \in S \implies$ c is an upper bound for S $(*1)$

(2): $c \leq b \quad \forall b \in B \implies$ c is the least of all upper bounds $(*2)$

Therefore from $(*1)$ and $(*2)$ we can say c is the least upper bound

2) Proof is similar to above

$\blacksquare$

Remarks about Supremum and Infimum

1) The supremum of a set S is the minimum of upperbounds

Let B be the set of all all upper bounds of S. Then

$$\boxed{\sup(S) = \min(B)}$$

Therefore since $\sup(S)$ is the minimum of a set, it is unique

2) If a set S has a maximum,

$$\boxed{\max(S) = \sup(S)}$$

To see this, let $b = \max(S)$, so b is an upperbound.

Any other $b' < b$ is not an upperbound $\implies b = \sup(S)$

3) Similarly if a set has a minimum,

$$\boxed{\min(S) = \inf(S)}$$

Equivalent Formulation of Infimum and Supremum

Formulations for Suprema

(Let $b = \sup(S)$)

1) If $b' < b$, then b' is **not** an upperbound
2) If $b' < b$, then $\exists x \in S$ such that $x > b'$
3) $\boxed{\text{If } \varepsilon > 0,\ \exists x \in S \text{ such that } x > b - \varepsilon}$

(Changing notation from b' to $b-\varepsilon$; $b' < b$ is equivalent to $\varepsilon > 0$)

Formulations for Infima

(Let $a = \inf(S)$)

1) If $a < a'$, then a' is **not** a lowerbound
2) If $a < a'$ then $\exists x \in S$ such that $x < a'$
3) $\boxed{\text{If } \varepsilon > 0,\ \exists x \in S \text{ such that } x < a + \varepsilon}$

(changing notation from a' to $a+\varepsilon$; $a' > a$ is equivalent to $\varepsilon > 0$

Infinite Spaces

We are going to look at metric spaces with infinite dimensions

Take the set $\mathbb{R}^n$, $\mathbb{R}^n$ is a finite dimensional object

Definition: Set $\mathbb{R}^{\mathbb{N}}$

The set $\mathbb{R}^{\mathbb{N}}$ is the set of sequences with real numbers as their entries

if $\underline{x} \in \mathbb{R}^{\mathbb{N}}$, then $\underline{x} = (x_1, \ldots, x_n, \ldots)$

We will try and put a metric on this:

$$\boxed{d_1(\underline{x}, \underline{y}) = \sum_{i=1}^{\infty} |x_i - y_i|}$$

and we need to check for convergence

Using this function to calculate the distance between $\underline{x}$ and $\underline{0} \in \mathbb{R}^{\mathbb{N}}$

$$d_1(\underline{x}, \underline{0}) = d_1((x_1, x_2, \dots), (0, 0, \dots))$$
$$= \sum_{i=1}^{\infty} |x_i - 0|$$
$$= \sum_{i=1}^{\infty} |x_i|$$

But this series may not converge. Hence not a real number. The value of the metric must be a real number.

Therefore we define a set ℓ_1

Definition: Set ℓ_1

ℓ_1 is the set of all numbers that satisfy

$$\sum_{i=1}^{\infty} |x_i| < \infty \qquad (*)$$

Therefore ℓ_1, the set of all real numbers satisfying $(*)$ AND

$$d(\underline{x}, \underline{y}) = \sum_{i=1}^{\infty} |x_i - y_i|$$

is a metric

Metric on ℓ_∞

Definition: Set ℓ_∞

ℓ_∞ is the set of all bounded sequences of real numbers

$$\underline{x} \in \ell_\infty \Rightarrow \exists\, M = M(x) > 0 \text{ such that } |x_i| \leq M \quad \forall i \in \mathbb{N}$$

where $\underline{x} = (x_1, \dots, x_n, \dots)$

Putting a metric on this, define

$$d_\infty(x, y) = \sup\{|x_i - y_i| : i \in \mathbb{N}\}$$

Space of all bounded sequences

X : set of all bounded sequences.

$(x_i)_{i\geq 1} \in X \implies \sup_i |x_i| < \infty$

Put metric $d(x,y)$ such that

$$\forall x,y \in X, \quad d(x,y) = \sup_i |x_i - y_i| = \sup\{|x_i - y_i| : i \in \mathbb{N}\}$$

$x = (x_i)_{i\geq 1}$

$y = (y_i)_{i\geq 1}$

Proving triangle inequality

if $x = (x_i)_{i\geq 1}$, $y = (y_i)_{i\geq 1}$, $z = (z_i)_{i\geq 1}$, $x, y, z \in X$,

$$\begin{aligned} |x_i - y_i| &\leq |x_i - z_i| + |z_i - y_i| \\ &\leq \sup_i |x_i - z_i| + \sup_i |z_i - y_i| \\ &= d(x,z) + d(z,y) \qquad \forall i \in \mathbb{N} \end{aligned}$$

$$\implies d(x,y) = \sup_i |x_i - y_i| \leq d(x,z) + d(z,y)$$

The space ℓ_p

Definition: ℓ_p set

Let X be the set of all sequences $x = (x_i)_{i\geq 1}$ such that

$$\left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p} < \infty \qquad p \geq 1$$

Define metric on X:

if $x = \{x_i\}_{i\geq 1}$, $y = \{y_i\}_{i\geq 1}$, then

$$d(x,y) = \left(\sum_{i=1}^{\infty} |x_i - y_i|^p\right)^{1/p}$$

This can be shown to satisfy axioms using minkowski inequality (infinite sum version)

Function Spaces

Applying our theory so far to function spaces

$$\boxed{\mathcal{F}(X,Y)}$$

where $\mathcal{F}(X,Y)$ is the set of all functions $f: X \to Y$

Lets consider 2 such examples

Definition: Function space $C([a,b])$

The function space $C([a,b])$ is the set of all continuous functions

$$f: [a,b] \to \mathbb{K}$$

$\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$

Definition Function space $B(S)$

The function space $B(S)$ is the set of all bounded functions

$$f: S \to \mathbb{K}$$

$\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$

Space of Bounded functions

Let $S \neq \emptyset$ (non-empty).

$B(S)$: the set of all bounded functions

$$f \in B(S) \iff \sup_{x \in S} |f(x)| < \infty$$

It follows that

$$f, g \in B(S) \iff \exists\ M > 0 \text{ and } N > 0 \text{ s.t}$$

$$\sup_{x \in S} |f(x)| \leq M \quad \text{and} \quad \sup_{x \in S} |g(x)| \leq N$$

Define metric (uniform metric on $B(S)$)

$$\boxed{d_\infty(f,g) = \sup_{x \in S} |f(x) - g(x)| \qquad \forall f, g \in B(S)}$$

Proving triangle inequality (M4) $f(x)$ is a real number, f is a function

$$|f(x)-g(x)| \le |f(x)-h(x)| + |h(x)-g(x)|$$
$$\le \sup_{x\in S}|f(x)-g(x)| + \sup_{x\in S}|h(x)-g(x)|$$
$$= d_\infty(f,h) + d_\infty(h,g) \qquad \forall f,g,h \in B(S)$$

$\Rightarrow \quad d_\infty(f,g) \le d_\infty(f,h) + d_\infty(h,g) \quad \forall f,g,h\in B(S)$

Space of Continuous functions

Consider the space of all continuous functions on interval $[a,b]$,

$C([a,b])$: The set of all continuous functions

$$f\in C([a,b]) \iff f:[a,b]\to\mathbb{R} \text{ is continuous}$$

The uniform metric on $C[a,b]$ is

$$\forall f,g\in C[a,b],\quad d_\infty(f,g) = \sup_{x\in[a,b]}|f(x)-g(x)|$$

More metrics on $C([a,b])$

Define d_1, d_2 metric as

$$\boxed{d_1(f,g) = \int_a^b |f(x)-g(x)|\,dx}$$

$$\boxed{d_2(f,g) = \left(\int_a^b |f(x)-g(x)|^2\,dx\right)^{1/2}}$$

Metric Spaces induced by Norm

Here we will define metric spaces on vector spaces

Basics of Vector Spaces

Definition: Vector Spaces

A vector space is a non-empty subset V of elements called vectors which satisfy

$$\mu \underline{u} + \lambda \underline{v} \in V \qquad \text{if } \underline{u}, \underline{v} \in V, \quad \lambda, \mu \in \mathbb{R}$$

where $\underline{u}, \underline{v}$ are vectors, μ, λ are scalars and $\mathbb{R}$ is the scalar field

Therefore we can abstract the notion of $a-b$ as we have:

$$\underline{u} - \underline{v} = \underline{u} + (-1)\underline{v} \qquad (\mu = 1, \lambda = -1)$$

We are missing an abstraction for the absolute value $|\cdot|$. The vector space version of the absolute value is called norm

Norm in a Vector Space

Definition: Norm

A function
$$\|\cdot\| : V \longrightarrow [0, \infty)$$

is a norm if it satisfies the following axioms

(N1) $\|\underline{v}\| \geq 0$

(N2) $\|\underline{v}\| = 0 \iff \underline{v} = 0$

(N3) $\|\lambda \underline{v}\| = \lambda \|\underline{v}\|$ where $\lambda \in \mathbb{R}$

(N4) $\|\underline{u} + \underline{v}\| \leq \|\underline{u}\| + \|\underline{v}\|$

Metric Space Induced by the Norm

We call a vector space with a norm a normed space

All normed spaces have a natural metric induced by the norm

$$d : V \times V \longrightarrow [0, \infty);$$
$$\|\underline{u} - \underline{v}\| = \|\underline{u} + (-1)\underline{v}\|$$

Example of vector space → normed space → metric space chain

Looking at $C([0,1])$, the set of all continuous functions on closed and bounded sets:
$C([0,1])$ is a vector space;

$$f:[0,1] \to \mathbb{R} \quad \text{and} \quad g:[0,1] \to \mathbb{R}$$

are vectors because

$$f+g:[0,1] \to \mathbb{R}\,;\ t \mapsto f(t)+g(t)$$

is a continuous function. Further taking a scalar value $\lambda \in \mathbb{R}$,

$$\lambda f:[0,1] \to \mathbb{R}\,;\ t \mapsto \lambda f(t)$$

is a continuous function. Therefore we can put norms $\|\cdot\|$ on $C([0,1])$

Note Going back to $\mathbb{R}^2$, we can define metric spaces d_1, d_2 and d_∞:

1) $d_1(x,y) = \sum_{i=1}^{n} |x_i - y_i|$

2) $d_2(x,y) = \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2}$

3) $d_\infty(x,y) = \max\{|x_i - y_i| : i \in \{1,\dots,n\}\}$

In analogue to this, we construct 3 norms $\|f\|_1$, $\|f\|_2$, $\|f\|_\infty$ which will induce a metric which will induce analogues to d_1, d_2, d_∞

- $\|f\|_1 = \int_0^1 |f(t)|\, dt$
- $\|f\|_2 = \left(\int_0^1 |f(t)|^2\right)^{1/2}$
- $\|f\|_\infty = \sup\{|f(t)| : t \in [0,1]\}$

Note We need to use integrals as input to the distance functions as we are dealing with functions hence continuous, not discrete but continuous hence sums gets reduced to integrals

Our corresponding induced metrics are

- $d_1(f,g) = \|f-g\|_1 = \int_0^1 |f(t)-g(t)|\, dt$
- $d_2(f,g) = \|f-g\|_2 = \left(\int_0^1 |f(t)-g(t)|^2\right)^{1/2}$

$d_\infty(f,g) = \|f-g\|_\infty = \sup\{|f(t)-g(t)| : t \in [0,1]\}$

Example of using metric induced by norm: d_2

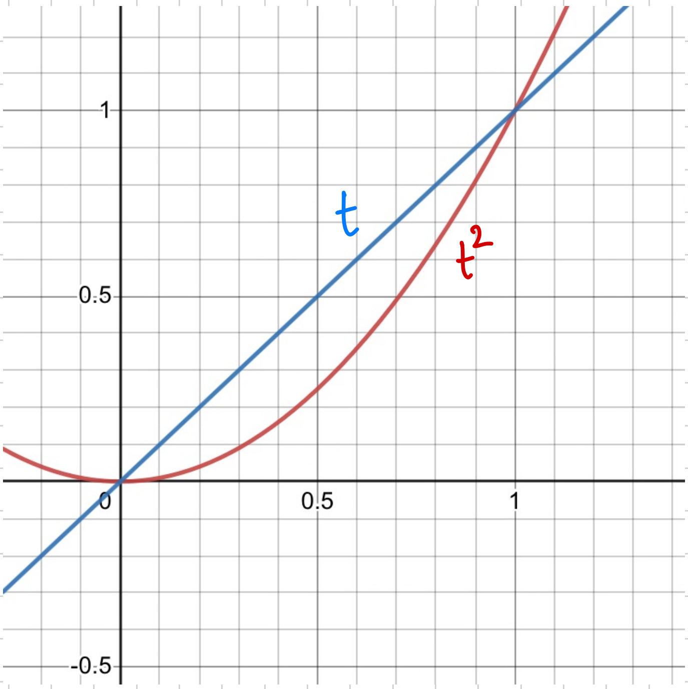
For example taking $f(t) = t$ and $g(t) = t^2$

$$d_2^2(f,g) = \int_0^1 |t^2 - t|^2 \, dt$$

$$= \int_0^1 (t - t^2) \, dt$$

$$= \int_0^1 (t^2 - 2t^3 + t^4) \, dt$$

$$= \left[\frac{t^3}{3} - \frac{t^4}{2} + \frac{t^5}{5} \right]_0^1 = \frac{1}{30}$$



Note

From the graph we can see that for

$$t \in [0,1]$$

we have that

$$t > t^2$$

and therefore we have

$$|t^2 - t| = (t - t^2) \geq 0$$

Generalizing C([0,1])

We can generalize $C([0,1])$ by $a, b \in \mathbb{R}$

$$C([a,b])$$

as long as $-\infty < a < b < \infty$

Furthermore we can replace $\mathbb{R}$ in $C([0,1])$ with $\mathbb{C}$: $f : [0,1] \to \mathbb{C}$

An important proposition

Below is an important proposition:

Proposition

Let (X,d) be a metric space. Define $d' : X \times X \to \mathbb{R}$ by

$$d'(x,y) = \frac{d(x,y)}{1 + d(x,y)}$$

Then d' is a metric on X

Proof: Showing triangle inequality (M4)

Suppose $x, y, z \in X$. Then by defn of metric,

i) $d(x,y) \geq 0$ (iii) $d(z,y) \geq 0$

(ii) $d(x,z) \geq 0$ (iv) $d(x,y) \leq d(x,z) + d(z,y)$

Therefore we can derive the following

$$d'(x,z) + d'(z,y) = \frac{d(x,z)}{1 + d(x,z)} + \frac{d(z,y)}{1 + d(z,y)}$$

$$\geq \frac{d(x,z) + d(z,y)}{1 + d(x,z) + d(z,y)}$$ look at inequalities sheet

$$= \frac{1}{1 + \frac{1}{d(x,z) + d(z,y)}}$$

$$\geq \frac{1}{1 + \frac{1}{d(x,y)}}$$ triangle inequality

$$= \frac{d(x,y)}{1 + d(x,y)}$$

$\blacksquare$

Space of all sequences of real numbers

Let $x=(x_i)_{i=1}^{\infty}$ be an arbitrary sequence of real numbers.

$$\boxed{\begin{array}{l} X: \text{the space of all sequences of real numbers} \\ \qquad x \in X \iff x=(x_i)_{i\geq 1} \text{ is a sequence of real numbers} \end{array}}$$

On this space, define metric

$$\boxed{d(x,y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|x_i - y_i|}{1+|x_i-y_i|}}$$

Prove the triangle inequality: (M4)

$$\begin{aligned} d(x,y) &= \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|x_i - y_i|}{1+|x_i-y_i|} \\ &= \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|x_i - z_i + z_i - y_i|}{1+|x_i-z_i+z_i-y_i|} \\ &\leq \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|x_i - z_i|}{1+|x_i-z_i|} + \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|x_i - y_i|}{1+|x_i-y_i|} \\ &= d(x,z) + d(z,y) \end{aligned}$$

Subspaces

Definition: Subspaces

Let (X,d) and $(Y,\hat{d})$ be metric spaces. We say $(Y,\hat{d})$ is a subspace of (X,d) if

1) $Y \subseteq X$
2) $\hat{d} = d|Y \times Y$ (restriction)

So all axioms of metric spaces hold in $(Y,\hat{d})$ so $(Y,\hat{d})$ is a metric space

Note: Observe the following facts

1) $d' = d|Y \times Y$ means that the distance function d' is restricted to the set $Y \times Y$ and can also be denoted by
$$\boxed{d|_Y}$$
2) We call X the ambient space

Note: Subspace does NOT mean subset

if (X,d) and Y a subspace, $Y^c = \phi$

Y can't see beyond itself

Examples of Subspaces

Example: Subspace Example 1

Take $(\mathbb{R},d)$ and take the subset $A = [0,1] \subseteq \mathbb{R}$

Here the metric is $d(x,y) = |x-y| \quad \forall x,y \in \mathbb{R}$

Consider the set
$$\Lambda = \{x \in \mathbb{R} \mid d(x,y) < 1/2\} = \left(\tfrac{1}{2}, \tfrac{3}{2}\right)$$

Now restricting the distance function to $A=[0,1]$ and using distance function $d|_A$ the points lying whithin $1/2$ are
$$\Lambda = \{x \in A : d|_A(x,1) < 1/2\} = \left(\tfrac{1}{2}, 1\right)$$

Isometrics

Definition: Isometry (ver 1)

Let (X,d) and $(Y,\hat{d})$ be metric spaces. An isometry onto $A \subseteq Y$ is a function

$$\psi : X \to Y$$

such that $\psi(X) = A$ (surjective onto A) and

$$d(x,y) = \hat{d}(\psi(x), \psi(y)) \quad \forall\, x,y \in X$$

we say that the metric space is isometric to subspace A

If $\psi(X) = Y$, that $A = Y$ then metric spaces X and Y are isometric

Definition: Isometry (ver 2)

Let (X,d) and $(Y,\hat{d})$ be metric spaces. They are said to be metrically equivalent or isometric if there are inverse functions

$$f : X \to Y \quad \text{and} \quad g : Y \to X$$

such that $\forall\, y,x \in X$

$$d(x,y) = \hat{d}(f(x), f(y))$$

and $\forall\, u,v \in Y$,

$$\hat{d}(u,v) = d(g(u), g(v))$$

In this event, we say the metric equivalence or isometry is defined on f and g

Examples of Isometric Equivalence

Example: Example of isometric equivalence

Consider the following functions which are examples of isometries

$$\psi : \mathbb{R} \to \mathbb{C} : t \mapsto t + 0i \qquad \psi^* : \mathbb{R} \to \mathbb{C} : t \mapsto t$$

$\mathbb{R}^2$ is isometric to $\mathbb{C}$ as define isometry

$$\psi : \mathbb{R}^2 \to \mathbb{C} \,;\ (x,y) \mapsto x + iy$$

$$|(x+iy) - (x'+iy')| = |(x-x') + i(y-y')| = \sqrt{(x-x')^2 + (y-y')^2}$$

distance function

$$= |(x,y) - (x',y')| \implies d'(\psi(a), \psi(b)) = d(a,b)$$

where $a = (x,y)$ $\quad b = (x',y')$